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Abstract

Rational J-inner-valued functions which are J-inner with respect to the unit circle (J being a matrix which is both self-adjoint and unitary) play an important role in interpolation theory and are extensively utilized in signal processing for filtering purposes and in control for minimal sensitivity (H_∞ feedback). Any such function is a product of three kinds of elementary factors, each of them having a unique singularity outside the unit disk, inside the unit disk and on the unit circle, respectively. Counterparts of the first kind have already been studied in the context of non-stationary systems, when analytic functions are replaced by upper triangular operators. The purpose of the present work is to study the non-stationary analogues of the factors of the third kind. One main difficulty is that one leaves the realm of bounded upper triangular operators and considers unbounded operators. Yet, as is the case for a number of special clases of non-stationary systems, all the systems under consideration are finitely specified, and the computations are done recursively on a finite set of state space data. We consider the particular case, where the operator given is of the IVI type (that is, it is time-invariant both for small and large indices, and is time-varying in between). The theory results in a rather general factorization theorem that generalizes the time-invariant case to finitely specified, time-varying systems. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let us first recall that the classical inverse scattering problem consists in finding all representations

$$s := T_{\Theta}(\sigma) = (\theta_{11}\sigma + \theta_{12})(\theta_{21}\sigma + \theta_{22})^{-1}$$

of a given function s analytic and contractive in the open unit disk $\mathbb{D}$ (a *Schur function*), where σ is still a Schur function and where

$$\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

is meromorphic in $\mathbb{D}$ and J-contractive: $\Theta(z)J\Theta(z)^* \leqslant J$ for all z in $\mathbb{D}$, where Θ is analytic, with

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The inequality means that the difference $J - \Theta(z)J\Theta(z)^*$ is a positive semidefinite matrix. If it also holds that $\Theta(z)J\Theta(z)^* = J$ a.e. on $\mathbb{T}$, it is called J-inner. The inverse scattering problem is closely related to the theory of linear time-invariant dissipative systems, and has numerous ramifications (see [1] for a survey). Two keystones in obtaining such Θ are the works of Schur in 1917 (the celebrated Schur algorithm; see [18]) and of Brune in 1930s (see [6, p. 14]). The resulting elementary (that is, of McMillan degree 1) Θ's are of the form

$$\Theta(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{(1-z)}{\epsilon \cdot (1-za^*)(1-a)} \begin{pmatrix} 1 & -k \\ k^* & -1 \end{pmatrix}. \tag{1.1}$$

In the case of the sections introduced by Schur (and later more generally by Nevanlinna), we have

$$a \in \mathbb{D}, \quad k \in \mathbb{D}, \quad \epsilon = \frac{1-|k|^2}{1-|a|^2},$$

while in the case of Brune sections, a and k are of modulus 1 (with $a \neq 1$) and ϵ is a strictly positive number. In the first case, Θ is also called a Blaschke factor (the matrix analogue of $(z-a)/(1-za^*)$) and its entries are bounded functions in the open unit disk. This boundedness property does not hold when a is on or outside the unit circle. In particular, Brune factors are not bounded in the open unit disk.

When one considers non-stationary systems, Schur functions are replaced by upper (or lower, depending on the convention) doubly infinite contractive matrices; see e.g., [4]. The analogue of the Blaschke factor is known (see e.g., [4]), but up to now, there was no known analogue of the Brune section. This is the problem we address in this paper. One of the main difficulties is that the definition of an upper triangular unbounded operator is not so clear in the present setting. For instance, if e_j is the canonical basis of $\ell_2(\mathbb{Z})$ and Z is the bilateral shift on $\ell_2(\mathbb{Z})$: $Ze_n = e_{n+1}$, then one has (in the weak topology and for all j)

$$\left(\sum_0^\infty Z^n\right)(e_j - e_{j-1}) = -\sum_{-\infty}^{-1} Z^n(e_j - e_{j-1}),$$

where the operator on the left is unbounded and "upper triangular" while the operator on the right is unbounded and "lower triangular". These operators are of course the analogues of the two Laurent expansions of $1/(1-z)$ centred at the origin. To remedy that difficulty, we will use the Zadeh extension (for the definition, see Section 3).

The outline of the paper is as follows: the paper consists of eight sections including this introduction. We review in Section 2, the notions of Blaschke and Brune factors in the case of analytic functions. A common formula defines both factors, although some of their properties are fundamentaly different. In Section 3, we review the non-stationary setting and recall the notion of Blaschke factor. The formula we use is taken from [8]. In contrast with the discrete case, this formula cannot be used right away to define Brune sections and we first introduce the Zadeh extension. Brune sections are studied in Section 4, and the non-stationary counterparts of points of local losslessness are studied in Section 5. Section 6 is devoted to a reproducing kernel approach to these problems. Section 7 deals with the question of factorization of the non-stationary (unbounded) J-inner functions and some concluding remarks are given in Section 8.

Part of the results presented in this paper has been announced in [2].

2. The stationary case

In this section, we recall how Blaschke factors and Brune sections appear in the inverse scattering problem. The section is given to provide motivation for the analysis in the non-stationary setting.

2.1. J-inner rational functions

For simplicity we first focus on the scalar case and recall the following one-dimensional version of a general structure theorem. The general case is given in Theorem 2.2.

Theorem 2.1. *Let $a, k \in \mathbb{C}$ and let κ be a strictly positive number. Let $\mathscr{M}$ be the one-dimensional Hilbert space spanned by the function*

$$F(z) = \begin{pmatrix} 1 \\ k^* \end{pmatrix} \Big/ (1 - za^*), \tag{2.1}$$

endowed with the norm $\|F\|_{\mathscr{M}} = \sqrt{\kappa}$. *Then the reproducing kernel of* $\mathscr{M}$ *is of the form*

$$\frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} \tag{2.2}$$

for some J-inner rational function Θ *if and only if*

$$\kappa(1 - |a|^2) = 1 - |k|^2. \tag{2.3}$$

The function Θ *is defined uniquely up to a multiplicative J-unitary constant on the right.*

It follows from (2.3) that κ can be chosen arbitrarily if a is on the unit circle and it has to be equal to $(1 - |k|^2)/(1 - |a|^2)$ otherwise.

The function Θ can be chosen to be normalized such that

$$\Theta(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

at any pre-assigned point $\mu \neq a$ on $\mathbb{T}$. Then

$$\Theta(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\mu - z}{(1 - za^*)(\mu - a)\kappa} \begin{pmatrix} 1 & k \\ k^* & 1 \end{pmatrix}, \tag{2.4}$$

where κ is a solution of (2.3).

Note that formula (2.4) follows from

$$\frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} = \frac{\begin{pmatrix} 1 \\ k^* \end{pmatrix} \begin{pmatrix} 1 & k \end{pmatrix}}{\kappa(1 - za^*)(1 - w^*a)} = \frac{F(z)F(w)^*}{\kappa} \tag{2.5}$$

with $w = \mu$ and $\Theta(\mu) = I_2$.

Let us briefly discuss Eq. (2.3) and formula (2.4). If $a \notin \mathbb{T}$, κ is uniquely defined and is equal to $(1 - |k|^2)/(1 - |a|^2)$. When $a \in \mathbb{D}$, we also have $k \in \mathbb{D}$ (since $\kappa > 0$) and Θ is a Blaschke factor (also called the Potapov factor of the first kind) and, after multiplication by an appropriate J-unitary constant on the right, it can also be written in the more familar form

$$\Theta_a(z) = \frac{1}{\sqrt{1 - |k|^2}} \begin{pmatrix} 1 & k^* \\ k & 1 \end{pmatrix} \begin{pmatrix} \frac{z-a}{1-za^*} & 0 \\ 0 & 1 \end{pmatrix}.$$

On the other hand, when $a \in \mathbb{T}$, Eq. (2.3) has a solution if and only if $k \in \mathbb{T}$. Then any $\kappa > 0$ is a solution (in fact, any $\kappa \in \mathbb{C}$ but these are not relevant for our exposition), and the corresponding Θ is a Brune section (also called Potapov factor of the third kind; the factors of the second kind correspond to the case, where $|a| > 1$).

A general characterization of rational J-inner functions has been given in [5]. In the statement and throughout the paper I_n stands for the identity $n \times n$ matrix.

Theorem 2.2. *Let $(C, A) \in \mathbb{C}^{2m\times n} \times \mathbb{C}^{n\times n}$ be an observable pair: $\bigcap_0^\infty \ker CA^j = \{0\}$, and let $\mathbb{P}$ be a strictly positive matrix. Let $F(z) = C(I_n - zA)^{-1}$ and let $\mathscr{M}$ be the vector space spanned by the columns of F with the inner product*

$$\left[F(z)c, F(z)d\right]_{\mathscr{M}} = d^*\mathbb{P}c.$$

Then $\mathscr{M}$ is a reproducing kernel Hilbert space and its reproducing kernel is of the form

$$\frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*}, \quad J = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}$$

for a J-inner rational function Θ if and only if $\mathbb{P}$ is a solution of the Stein equation

$$\mathbb{P} - A^*\mathbb{P}A = C^*JC. \tag{2.6}$$

In the latter case Θ may be chosen to be normalized to I_{2n} at a pre-assigned point $\mu \neq a$ on $\mathbb{T}$:

$$\Theta(z) = I_{2m} - (1 - \mu^*z)C(I_n - zA)^{-1}\mathbb{P}^{-1}(I_n - \mu A)^{-*}C^*J \tag{2.7}$$

and is defined uniquely up to a right J-unitary factor.

This result is the finite-dimensional version of a theorem of de Branges. We present an analogue of this theorem in the non-stationary setting in the sequel; see Theorem 6.1.

We have the formulas:

$$C(I_n - zA)^{-1}\mathbb{P}^{-1}(I_n - wA)^{-*}C^* = \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} \tag{2.8}$$

and

$$\begin{aligned}\Theta(z) = {} & I - C\mathbb{P}^{-1}(I_n - \mu A)^{-1}C^*J \\ & + zC(\mu^*I_n - A)(I_n - zA)^{-1}\mathbb{P}^{-1}(I_n - \mu A)^{-*}C^*J.\end{aligned} \tag{2.9}$$

Furthermore, the matrix

$$\begin{pmatrix} A & \mathbb{P}^{-1}(I_n - \mu A)^{-*}C^*J \\ C(\mu^*I_n - A) & I - C\mathbb{P}^{-1}(I_n - \mu A)^{-1}C^*J \end{pmatrix} \tag{2.10}$$

is

$$\begin{pmatrix} \mathbb{P} & 0 \\ 0 & J \end{pmatrix}\text{-unitary.}$$

So, the reproducing kernel Hilbert space with reproducing kernel of the form (2.2) is really determined by the first block column of the matrix (2.10).

2.2. Points of local losslessness

We recall the following theorem:

Theorem 2.3. *Let $a, k \in \mathbb{D}$ and let Θ be the corresponding Blaschke factor. The formula $s = T_\Theta(\sigma)$ describes the set of all Schur functions s such that $s(a) = k$ when σ runs through the family of all Schur functions.*

For the non-stationary analogue of this result, see Theorem 3.2.

Theorem 2.3 shows that it is possible to "extract" a Blaschke factor at any interior point (at least in the scalar case). A main difference with Brune sections is that it is not possible to extract a Brune section at any boundary point. One obvious reason for that is that the given Schur function need not have a non-tangential limit at a given point on $\mathbb{T}$, but the whole story is more subtle, as we now recall. Let us start with a Schur function s, fix a ($|a| = 1$) and apply Theorem 2.3 to the point $\rho a \in \mathbb{D}$ and to $k = s(\rho a)$ with $0 \leqslant \rho < 1$. The Blaschke section is equal to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{(1-z)}{\frac{1-|s(\rho a)|^2}{1-\rho^2} \cdot (1 - z\rho a^*)(1-\rho a)} \begin{pmatrix} 1 & -s(\rho a) \\ s(\rho a)^* & -1 \end{pmatrix}. \tag{2.11}$$

Proposition 2.4. *The pointwise limit* $\lim_{\rho \to 1} \Theta(\rho, z)$ *exists for all* $z \in \mathbb{D}$ *and is not identically equal to the identity if and only if the following two conditions hold*:
1. *The limit* $\lim_{\rho \to 1} s(\rho a)$ *exists and is unitary.* (*We will denote the limit by* $s(a)$.)
2. *The limit* $\lim_{\rho \to 1} (1 - |s(\rho a)|^2)/(1-\rho^2)$ *exists and is strictly positive.*

Definition 2.5. Let s be a Schur function. A point $a \in \mathbb{T}$ for which the conditions of the previous proposition hold is called a *point of local losslessness* for s.

The main problem we address in this paper is, as already mentioned, the study of the analogues of Brune sections in the non-stationary case.

2.3. The interpolation problem associated to a Brune section

There is no straightforward analogue of Theorem 2.3 for Brune section.

Theorem 2.6. *Let $\mu,\ a,\ k \in \mathbb{T}$, let $p > 0$ and let $\Theta(z)$ be a Brune section defined in* (2.4) *and normalized to I_2 at the point $\mu \neq a$. Then the formula $s = T_\Theta(\sigma)$ describes the set of all Schur functions s such that*

$$\lim_{\rho \to 1} s(\rho a) = k \quad \textit{and} \quad \lim_{\rho \to 1} \frac{1 - |s(\rho a)|^2}{1 - \rho^2} \leqslant p. \tag{2.12}$$

Proof. It was shown in [17] that s is a Schur function and satisfies conditions (2.12) if and only if the following inequality

$$\begin{pmatrix} p & \frac{1-s(z)k^*}{1-za^*} \\ \frac{1-ks(z)^*}{1-az^*} & \frac{1-|s(z)|^2}{1-|z|^2} \end{pmatrix} \geqslant 0$$

holds at every point $z \in \mathbb{D}$. Using the Schur complement one can conclude that the last inequality is equivalent to

$$1 - |s(z)|^2 - (1 - |z|^2)\frac{|1 - s(z)k^*|^2}{|1 - za^*|^2 p} \geqslant 0.$$

Representing the latter inequality in the form

$$(1, -s(z))J\left(1\ s(z)^*\right) - (1, -s(z))\frac{(1 - |z|^2)}{|1 - za^*|^2 p}\begin{pmatrix}1\\k\end{pmatrix}(1, k^*)\begin{pmatrix}1\\-s(z)^*\end{pmatrix} \geqslant 0$$

and taking (2.5) (with $z = w$) into account, we obtain

$$(1, -s(z))\Theta(z)J\Theta(z)^*\begin{pmatrix}1\\-s(z)^*\end{pmatrix} \geqslant 0. \tag{2.13}$$

Thus, s is a Schur function and satisfies conditions (2.12) if and only if (2.13) holds at every point $z \in \mathbb{D}$. Since Θ is J-inner, the latter is equivalent (for the proof see e.g., [11]) to a representation $s = T_\Theta(\sigma)$ of s for some Schur function σ. □

Note that Theorem 2.6 was proven in [17] for matrix valued Schur functions and a point of local losslessness of higher order. For further tangential and multipoint generalizations see [15,16]. The proof of Theorem 2.7 is in much the same as the proof of Theorem 2.6. In what follows the set of all $p \times q$ Schur functions is denoted by $\mathscr{S}^{p\times q}$.

Theorem 2.7. *Let μ, $a \in \mathbb{T}$, let $\mathbb{P} \in \mathbb{C}^{n\times n}$ be a strictly positive matrix, let*

$$C = \begin{pmatrix}C_1\\C_2\end{pmatrix} \in \begin{pmatrix}\mathbb{C}^{p\times n}\\\mathbb{C}^{q\times n}\end{pmatrix} \quad \textit{be such that} \quad C^*JC = C_1^*C_1 - C_2^*C_2 = 0 \tag{2.14}$$

and let $\Theta(z)$ be a Brune section defined in (2.7) and normalized to I_2 at the point $\mu \neq a$. Then the formula $S = T_\Theta(\sigma)(\sigma \in \mathscr{S}^{p\times q})$ describes the set of all Schur functions $S \in \mathscr{S}^{p\times q}$ such that

$$\widetilde{C} := \lim_{\rho\to 1} C_1^* S(\rho a) = C_2^* \quad \textit{and} \quad \widetilde{\mathbb{P}} := \lim_{\rho\to 1} C_1^*\frac{I_p - S(\rho a)S(\rho a)^*}{1 - \rho^2}C_1 \leqslant \mathbb{P}. \tag{2.15}$$

Let $S \in \mathscr{S}^{p\times q}$ be a given Schur function and let $a \in \mathbb{T}$ and $C_1 \in \mathbb{C}^{p\times n}$. We shall say that Θ of the form (2.7) (for some choice of $\mathbb{P}$ and C_2) is a (C_1, a)-solution of the lossless inverse scattering problem if $S = T_\Theta(\sigma)$ for some choice of $\sigma \in \mathscr{S}^{p\times q}$.

Theorem 2.8. *Let $S \in \mathscr{S}^{p\times q}$, let $a \in \mathbb{T}$ and let $C_1 \in \mathbb{C}^{p\times n}$ with* rank $C_1 = p \leqslant n$. *Then the function Θ of the form (2.7) is a (C_1, a)-solution of the lossless inverse scattering problem if and only if the limits $\widetilde{C}$ and $\widetilde{\mathbb{P}}$ in (2.15) exist and the parameters C_2 and $\mathbb{P}$ in the definition of Θ meet the conditions $C_2 = \widetilde{C}^*$ and $\mathbb{P} \leqslant \widetilde{\mathbb{P}}$.*

3. Some preliminaries

3.1. The non-stationary stationary setting

For the setting described here, we refer to [4,9]. We denote by $\mathcal{N} = \{\mathcal{N}_i\}$ a sequence of separable Hilbert spaces indexed by the integers, and by $\mathscr{X}(\ell^2_{\mathcal{N}})$ the set of bounded linear operators from the space $\ell^2_{\mathcal{N}}$ of square summable sequences with jth component in $\mathcal{N}_j$ into itself. We shall often drop the dependence on $\mathcal{N}$ and write $\mathscr{X}$. The space $\ell^2_{\mathcal{N}}$ is taken with the standard inner product. Let $Z_{\mathcal{N}}$ be the bilateral backward shift operator

$$(Z_{\mathcal{N}} f)_i = f_{i+1}, \quad i = \ldots, -1, 0, 1, \ldots,$$

where $f = (\ldots, f_{-1}, f_0, f_1, \ldots) \in \ell^2_{\mathcal{N}}$. The operator $Z_{\mathcal{N}}$ is unitary on $\ell^2_{\mathcal{N}}$, i.e., $Z_{\mathcal{N}} Z^*_{\mathcal{N}} = Z^*_{\mathcal{N}} Z_{\mathcal{N}} = I_{\mathcal{N}}$, and

$$\pi_0^* Z^j_{\mathcal{N}} \pi_0 = \begin{cases} I_{\mathcal{N}_0} & \text{if } j = 0, \\ 0_{\mathcal{N}_j} & \text{if } j \neq 0, \end{cases}$$

where π_0 denotes the injection map

$$\pi_0 : u \in \mathcal{N}_0 \to f \in \ell^2_{\mathcal{N}}, \quad \text{where } \begin{cases} f_0 = u, \\ f_i = 0, & i \neq 0. \end{cases}$$

We define the space of upper triangular operators by

$$\mathscr{U}\left(\ell^2_{\mathcal{N}}\right) = \left\{ A \in \mathscr{X}\left(\ell^2_{\mathcal{N}}\right) \,\middle|\, \pi^*_{\mathcal{N}} Z^i_{\mathcal{N}} A Z^{*j}_{\mathcal{N}} \pi_{\mathcal{N}} = 0 \ \text{ for } i > j \right\}$$

and the space of lower triangular operators by

$$\mathscr{L}\left(\ell^2_{\mathcal{N}}\right) = \left\{ A \in \mathscr{X}\left(\ell^2_{\mathcal{N}}\right) \,\middle|\, \pi^*_{\mathcal{N}} Z^i_{\mathcal{N}} A Z^{*j}_{\mathcal{N}} \pi_{\mathcal{N}} = 0 \ \text{ for } i < j \right\}.$$

The space of diagonal operators $\mathscr{D}(\ell^2_{\mathcal{N}})$ consists by definition of the operators which are both upper and lower triangular. We denote these spaces by $\mathscr{U}$, $\mathscr{L}$ and $\mathscr{D}$. Similarly, we write Z instead of $Z_{\mathcal{N}}$ and I instead of $I_{\mathcal{N}}$.

Letting $A^{(j)} = Z^{*j} A Z^j$ for $A \in \mathscr{X}$ and $j \in \mathbb{Z}$, note that $(A^{(j)})_{st} = A_{s-j,t-j}$ and that the maps $A \mapsto A^{(j)}$ take the spaces $\mathscr{L}$, $\mathscr{D}$, $\mathscr{U}$ into themselves. In [4] it was shown that for every $F \in \mathscr{U}$, there exists a unique sequence of operators $F_{[j]} \in \mathscr{D}$ $(j \geqslant 0)$ such that

$$F - \sum_{j=0}^{n-1} Z^j F_{[j]} \in Z^n \mathscr{U}.$$

In fact, $(F_{[j]})_{ii} = F_{i-j,i}$ and we can formally represent $F \in \mathscr{U}$ as the sum of its diagonals

$$F = \sum_{n=0}^{\infty} Z F_{[n]}.$$

We now define the left W-transform

$$F^{\wedge}(W) = \sum_{n=0}^{\infty} W^{[n]} F_{[n]} = \sum_{n=0}^{\infty} (WZ^*)^n Z^n F_{[n]}$$

with

$$W^{[0]} = I \quad \text{and} \quad W^{[j+1]} = W(W^{[j]})^{(1)} \quad \text{for } (j \geqslant 0)$$

for any $W \in \mathscr{X}$ for which

$$\ell_W = \lim_{n\uparrow\infty} \left\| W^{[n]} \right\|^{1/n} < 1,$$

where the last limit is the spectral radius $r_{\mathrm{sp}}(WZ^*)$ of WZ^*. This transform was introduced in [3].

We recall that $F^{\wedge}(W)$ is the unique diagonal operator D such that

$$(Z - W)^{-1}(F - D) \in \mathscr{U}.$$

The following theorem is proved in [8, Theorem 7.3, p. 212]. Before stating the theorem we recall that in [8, p. 160] the following class was introduced.

Definition 3.1. The operator matrix $\Theta = (\Theta_{ij})$ with $\Theta_{ij} \in \mathscr{U}$ belongs to the class $\mathscr{A}$ if Θ_{22} is invertible in $\mathscr{U}$ and if Θ is J-unitary, where

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Theorem 3.2. *Let $V_0 \in \mathscr{D}$ and such that $\ell_{V_0} < 1$. Moreover, let $\alpha, \beta \in \mathscr{D}$ be such that the sum*

$$\Lambda = \sum_{0}^{\infty} V_0^{[n]}(\alpha\alpha^* - \beta\beta^*)^{(n)} V_0^{[n]*} \tag{3.1}$$

converges in the operator norm to a strictly positive and invertible diagonal operator Λ. Then there exist diagonal operators q_1 and q_2 and an operator $\Theta = (\Theta)_{ij=1,2}$ defined by

$$\Theta_{11} = q_2 + \alpha^* Z(I - V_0^* Z)^{-1}(\Lambda^{-1/2})^{(1)} q_1, \tag{3.2}$$
$$\Theta_{12} = \alpha^*(I - ZV_0^*)^{-1}\Lambda^{-1}\beta(I + \beta^*\Lambda^{-1}\beta)^{-1/2}, \tag{3.3}$$
$$\Theta_{21} = \beta^* Z(I - V_0^* Z)^{-1}(\Lambda^{-1/2})^{(1)} q_1, \tag{3.4}$$
$$\Theta_{22} = (I + \beta^*(I - ZV_0^*)^{-1}\Lambda^{-1}\beta) \cdot (I + \beta^*\Lambda^{-1}\beta)^{-1/2} \tag{3.5}$$

such that $\Theta \in \mathscr{A}$ and the linear fractional transformation

$$S = (\Theta_{11}\sigma + \Theta_{12})(\Theta_{21}\sigma + \Theta_{22})^{-1} \tag{3.6}$$

describes the set of all contractive $S \in \mathscr{U}$ such that $(\alpha S)^{\wedge}(V_0) = \beta$.

The operator block matrix Θ defined by (3.12)–(3.15) is the analogue of the Blaschke factors (Potapov factor of the first kind). The operator Λ satisfies the equation

$$\Lambda - V_0\Lambda^{(1)}V_0^* = \alpha\alpha^* - \beta\beta^*. \tag{3.7}$$

We note that we can write

$$\Theta = D + CZ(I - ZA)^{-1}B, \tag{3.8}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is given by

$$\begin{pmatrix} (\Lambda^{1/2})^{(1)}V_0^*\Lambda^{-1/2} & q_1 & (\Lambda^{1/2})^{(1)}V_0^*\Lambda^{-1}\beta(I+\beta^*\Lambda^{-1}\beta)^{-1/2} \\ \alpha^*\Lambda^{-1/2} & q_2 & \alpha^*\Lambda^{-1}\beta(I+\beta^*\Lambda^{-1}\beta)^{-1/2} \\ \beta^*\Lambda^{-1/2} & 0 & (I+\beta^*\Lambda^{-1}\beta)^{1/2} \end{pmatrix} \tag{3.9}$$

with $\binom{q_1}{q_2}$ a vector of two block diagonals, whose columns form an orthonormal basis complementary, for each diagonal index k, k, to the space spanned by the columns of the block diagonal entries in

$$\begin{pmatrix} (\Lambda^{1/2})^{(1)}V_0^*\Lambda^{-1/2} \\ \alpha^*\Lambda^{-1/2} \end{pmatrix} \tag{3.10}$$

at the same diagonal index.

We also note that the operator block matrix (3.9) is $\widetilde{J}$-unitary, where

$$\widetilde{J} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix}. \tag{3.11}$$

It is tempting to put $V_0 \in \mathscr{D}$ with $\ell_{V_0} = 1$ in these formulas and call the result a Brune section with Λ being a solution of (3.7). Unfortunately, the operator $I - ZV_0^*$ is then non-invertible (note that it has a dense range). We will deal with this problem in the next section using the Zadeh extension of an operator.

In the special case, where each (diagonal) entry in α is square and non-singular, the formulas can be written in closed form:

$$\begin{aligned} \Theta_{11} = {} & \big(-\alpha^{-1}V_0\Lambda^{(1)} + \alpha^*(I - ZV_0^*)^{-1}Z\big) \\ & \times\big(\Lambda^{(1)} + \Lambda^{(1)}V_0^*(\alpha\alpha^*)^{-1}V_0\Lambda^{(1)}\big)^{-1/2}, \end{aligned} \tag{3.12}$$

$$\Theta_{21} = \beta^*Z(I - Z_0V^*)^{-1}\cdot\big(\Lambda^{(1)}\big)^{-1/2}\big(I + V_0^*(\alpha\alpha^*)^{-1}V_0\big)^{-1/2}, \tag{3.13}$$

$$\Theta_{12} = \alpha^*(I - ZV^*)^{-1}\Lambda^{-1}\beta\cdot(I+\beta^*\Lambda^{-1}\beta)^{-1/2}, \tag{3.14}$$

$$\Theta_{22} = (I + \beta^*(I - ZV^*)^{-1}\Lambda^{-1}\beta)\cdot(I+\beta^*\Lambda^{-1}\beta)^{-1/2}. \tag{3.15}$$

In that case the diagonal operators q_1 and q_2 are defined by

$$q_1 = (\Lambda^{1/2})^{(1)}\big(\Lambda^{(1)} + \Lambda^{(1)}V_0^*(\alpha\alpha^*)^{-1}V_0\Lambda^{(1)}\big)^{-1/2}, \tag{3.16}$$

$$q_2 = -\alpha^{-1}V_0\Lambda^{(1)}\big(\Lambda^{(1)} + \Lambda^{(1)}V_0^*(\alpha\alpha^*)^{-1}V_0\Lambda^{(1)}\big)^{-1/2}. \tag{3.17}$$

The Redheffer transform of $\Theta \in \mathscr{U}^{2\times 2}$ with an invertible block entry Θ_{22} is defined by

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} \Theta_{11} - \Theta_{12}(\Theta_{22})^{-1}\Theta_{21} & -\Theta_{12}(\Theta_{22})^{-1} \\ (\Theta_{22})^{-1}\Theta_{21} & (\Theta_{22})^{-1} \end{pmatrix}. \tag{3.18}$$

We note that $\Sigma \in \mathscr{U}^{2\times 2}$ and is unitary if $\Theta \in \mathscr{A}$.

Proposition 3.3. *Let Θ be the block operator matrix with block entries given by* (3.12)–(3.15), *let $\alpha\alpha^*$ be non-singular and let*

$$V = V_0^*(\Lambda + \beta\beta^*)^{-1}\Lambda. \tag{3.19}$$

Then $\ell_V < 1$, and the Redheffer transfrom of Θ is given by

$$\Sigma_{11} = q_2 + \alpha^*(\Lambda + \beta\beta^*)^{-1}\Lambda Z(I - VZ)^{-1}(\Lambda^{-1/2})^{(1)}q_1, \tag{3.20}$$

$$\Sigma_{12} = -\alpha^*(\Lambda + \beta\beta^* - \Lambda Z V_0^*)^{-1}\beta, \tag{3.21}$$

$$\Sigma_{21} = (I + \beta^*\Lambda^{-1}\beta)^{-1/2}\beta^* Z(I - VZ)^{-1}(\Lambda^{-1/2})^{(1)}q_1, \tag{3.22}$$

$$\Sigma_{22} = (I + \beta^*\Lambda^{-1}\beta)^{1/2} \cdot (I - \beta^*(\Lambda + \beta\beta^* - \Lambda Z V_0^*)^{-1}\beta). \tag{3.23}$$

Proof. We first prove that $\ell_V < 1$. Because of state equivalence, it is enough to verify that

$$\Delta \overset{\text{def.}}{=} (\Lambda^{1/2})^{(1)}V_0^*(\Lambda + \beta\beta^*)^{-1}\Lambda^{1/2} < 1.$$

This is done as follows:

$$\begin{aligned} I - \Delta^*\Delta &= I - \Lambda^{1/2}V_0^*(\Lambda + \beta\beta^*)^{-1}V_0^*\Lambda^{(1)}V_0(\Lambda + \beta\beta^*)^{-1}\Lambda^{1/2} \\ &= \Lambda^{1/2}(\Lambda + \beta\beta^*)^{-1}\big((\Lambda + \beta\beta^*)\Lambda^{-1}(\Lambda + \beta\beta^*) - V_0\Lambda^{(1)}V_0^*\big) \\ &\quad \times(\Lambda + \beta\beta^*)^{-1}\Lambda^{1/2} \\ &= \Lambda^{1/2}(\Lambda + \beta\beta^*)^{-1}\big(\Lambda + 2\beta\beta^* + \beta\beta^*\Lambda^{-1}\beta\beta^* - (\Lambda + \beta\beta^* - \alpha\alpha^*)\big) \\ &\quad \times(\Lambda + \beta\beta^*)^{-1}\Lambda^{1/2} \\ &= \Lambda^{1/2}(\Lambda + \beta\beta^*)^{-1}\big(\alpha\alpha^* + \beta(I + \beta^*\Lambda^{-1}\beta)\beta^*\big)(\Lambda + \beta\beta^*)^{-1}\Lambda^{1/2}, \end{aligned}$$

from which we conclude that Δ has the operator norm strictly less than 1, since $\alpha\alpha^*$ and hence

$$(\alpha\alpha^* + \beta(I + \beta^*\Lambda^{-1}\beta)\beta^*)$$

is assumed invertible.

The fact that Σ is unitary follows then from the fact that Θ is J-unitary. □

We note that we can write

$$\Sigma = D_\Sigma + C_\Sigma Z(I - A_\Sigma Z)^{-1} B_\Sigma, \tag{3.24}$$

where

$$\begin{pmatrix} A_\Sigma & B_\Sigma \\ C_\Sigma & D_\Sigma \end{pmatrix}$$

is given by

$$\begin{pmatrix} (\Lambda^{1/2})^{(1)} V_0^* (\Lambda + \beta\beta^*)^{-1} \Lambda^{1/2} & q_1 & -(\Lambda^{1/2})^{(1)} V_0^* \Lambda^{-1} \beta (I + \beta^* \Lambda^{-1} \beta)^{-1} \\ \alpha^* (\Lambda + \beta\beta^*)^{-1} \Lambda^{1/2} & q_2 & -\alpha^* \Lambda^{-1} \beta (I + \beta^* \Lambda^{-1} \beta)^{-1} \\ (I + \beta^* \Lambda^{-1} \beta)^{-1/2} \beta^* \Lambda^{-1/2} & 0 & (I + \beta^* \Lambda^{-1} \beta)^{-1/2} \end{pmatrix}, \tag{3.25}$$

where q_1 and q_2 are defined as above by (3.16) and (3.17), and is unitary. From these follows a more general expression for the entries of Σ than given in the proposition.

3.2. The Zadeh extension

The main point to be exploited in the sequel is that Σ makes sense even when $\ell_V = 1$, provided the operator

$$\Delta = (\Lambda^{1/2})^{(1)} V_0^* (\Lambda + \beta\beta^*)^{-1} \Lambda^{1/2} \tag{3.26}$$

is such that $\ell_\Delta < 1$. To define a corresponding operator matrix Θ, one cannot use anymore formulas (3.9), and one uses a device first introduced for bounded operators.

Definition 3.4. Let $S \in \mathscr{U}$ and let $S_{[n]}$ be the diagonal operators such that for all $n \in \mathbb{N}$, $(S - \sum_0^n Z^j S_{[j]}) \in Z^{n+1}\mathscr{U}$. The Zadeh extension of S is defined by

$$S(t) = \sum_0^\infty t^n Z^n S_{[n]}. \tag{3.27}$$

Lemma 3.5. *Let $S \in \mathscr{U}$ and $t \in \mathbb{D}$. Then $S(t) \in \mathscr{U}$ and*

$$S(t)^\wedge(W) = S^\wedge(tW) \tag{3.28}$$

for every $W \in \mathscr{D}$ with $\ell_W < 1$.

Dym and Freydin made extensive use of this extension; see [12,13]. They prove in particular the following theorem.

Theorem 3.6. *For* $U, V \in \mathscr{U}$,

$$(UV)(t) = U(t)V(t) \text{ and } \|U(t)\| \leqslant \|U\|, \quad t \in \mathbb{D}.$$

The following proposition will be used in the proof of Theorem 5.2 on the extraction of non-stationary Brune sections at points of local losslessness.

Proposition 3.7. *Let* σ_r *be a sequence of upper triangular contractions converging weakly to* σ *for* $r \to 1$. *Then* σ *is also upper triangular and*

$$\lim_{r\to 1} \sigma_r(t) = \sigma(t)$$

for every $t \in (0, 1)$, *where the limit is meant in the weak sense.*

Proof. Let $(\sigma_r)_{kj}$ and $(\sigma)_{kj}$ be the matrix representations of σ_r and σ, respectively. Then the matrix representations of $\sigma_r(t)$ and $\sigma(t)$ are $t^{k-j}(\sigma_r)_{kj}$ and $t^{k-j}(\sigma)_{kj}$, respectively. We have

$$\begin{aligned}\lim_{r\to 1}\langle\sigma_r(t)e_j, e_k\rangle &= t^{k-j}\lim_{r\to 1}\langle\sigma_r e_j, e_k\rangle\\ &= t^{k-j}\langle\sigma e_j, e_k\rangle\\ &= \langle\sigma(t)e_j, e_k\rangle.\end{aligned}$$

Since $\|\sigma_r(t)\| \leqslant 1$ and $\|\sigma(t)\| \leqslant 1$, the proposition follows. □

We note the following: given a (not necessarily uniformly bounded) sequence of diagonal operators $\mathbf{D} = (D_n)_0^\infty$ we can sometimes define

$$\mathbf{D}(t) = \sum_0^\infty t^n Z^n D_n \tag{3.29}$$

for $t \in [0, 1)$. In general $\mathbf{D}(t)$ is not the Zadeh extension of $\mathbf{D}(1)$, which need not exist as an operator in $\mathscr{U}$. Take for instance $D_n = I$ or $D_n = n \cdot I$.

We will also define the Zadeh extension for elements $U \in \mathscr{U}^{2\times 2}$ by

$$U(t) \stackrel{\text{def.}}{=} (U_{ij}(t))_{i,j=1,2}.$$

It is not difficult to see that Theorem 3.6 still holds in this case. In particular we have the following proposition.

Proposition 3.8. *The operator* Θ *defined by* (3.12)–(3.15) *satisfies*

$$\Theta(t)J\Theta(t)^* \leqslant J, \quad \Theta(t)^*J\Theta(t) \leqslant J, \quad t \in \mathbb{D}. \tag{3.30}$$

Proof. It suffices to notice that the Redheffer transform of the analytic function $t \mapsto \Theta(t)$ is contractive in the open unit disk, and the fact that the Redheffer transform is contractive if and only if the original function is *J*-contractive. □

4. Brune sections and more

4.1. Brune sections in state space form

To extend the previous analysis to possibly unbounded Θ, we need only to assume that the Stein equation (3.7) has a strictly positive and invertible solution. Then, the operator

$$\begin{pmatrix} (\Lambda^{1/2})^{(1)} V_0^* \Lambda^{-1/2} \\ \alpha^* \Lambda^{-1/2} \\ \beta^* \Lambda^{-1/2} \end{pmatrix} \tag{4.1}$$

is $\widetilde{J}$-isometric.

Define

$$a = (\Lambda^{1/2})^{(1)} V_0^* (\Lambda + \beta\beta^*)^{-1/2},$$
$$c = \alpha^* (\Lambda + \beta\beta^*)^{-1/2}.$$

Because of the Stein equation we have that

$$a^* a + c^* c = 1.$$

Theorem 4.1. *Let q_1 and q_2 be diagonal operators such that*

$$\begin{pmatrix} a & q_1 \\ c & q_2 \end{pmatrix}$$

form a unitary operator (*notice that none of them is necessarily square*). *Then the matrix defined by*

$$\begin{pmatrix} (\Lambda^{1/2})^{(1)} V_0^* \Lambda^{-1/2} & q_1 & (\Lambda^{1/2})^{(1)} V_0^* \Lambda^{-1} \beta (I + \beta^* \Lambda^{-1} \beta)^{-1/2} \\ \alpha^* \Lambda^{-1/2} & q_2 & \alpha^* \Lambda^{-1} \beta (I + \beta^* \Lambda^{-1} \beta)^{-1/2} \\ \beta^* \Lambda^{-1/2} & 0 & (I + \beta^* \Lambda^{-1} \beta)^{1/2} \end{pmatrix} \tag{4.2}$$

is $\widetilde{J}$-unitary.

Let

$$V = (\Lambda^{1/2})^{(1)} V_0^* (\Lambda + \beta\beta^*)^{-1} \Lambda^{1/2}. \tag{4.3}$$

Then $\ell_V < 1$ as soon as the diagonal operator

$$(\alpha\alpha^* + \beta(I + \beta^* \Lambda^{-1} \beta)\beta^*)$$

is boundedly invertible.

Proof. The proof that $\ell_V < 1$ is as in the proof of Proposition 3.3. Now $\ell_V < 1$ follows from the hypothesis made on $(\alpha\alpha^* + \beta(I + \beta^* \Lambda^{-1} \beta)\beta^*)$.

The second block-column is orthogonal to the first by definition, but it is also orthogonal to the last because the two top blocks of the last column are actually 'proportional' to the first block column since they are given by

$$\begin{pmatrix} (\Lambda^{1/2})^{(1)} V_0^* \Lambda^{-1/2} \\ \alpha^* \Lambda^{-1/2} \end{pmatrix} \Lambda^{-1/2}\beta(I+\beta^*\Lambda^{-1}\beta)^{-1/2}. \tag{4.4}$$

The corresponding realization for the Redheffer transform is the same as before, and is given by utilizing the transformation described above

$$\begin{pmatrix} (\Lambda^{1/2})^{(1)} V_0^*(\Lambda+\beta\beta^*)^{-1}\Lambda^{1/2} & q_1 & -(\Lambda^{1/2})^{(1)} V_0^* \Lambda^{-1}\beta(I+\beta^*\Lambda^{-1}\beta)^{-1} \\ \alpha^*(\Lambda+\beta\beta^*)^{-1}\Lambda^{1/2} & q_2 & -\alpha^*\Lambda^{-1}\beta(I+\beta^*\Lambda^{-1}\beta)^{-1} \\ (I+\beta^*\Lambda^{-1}\beta)^{-1/2}\beta^*\Lambda^{-1/2} & 0 & (I+\beta^*\Lambda^{-1}\beta)^{-1/2} \end{pmatrix}. \tag{4.5}$$

The corrersponding Σ is inner (that is, a unitary and causal operator). Unfortunately,

$$\ell_{(\Lambda^{1/2})^{(1)} V_0^* \Lambda^{-1/2}} = \ell_{V_0} = 1$$

and so we cannot in general define Θ via the operator matrix (4.2) as in (3.8). □

The forms obtained for Θ are now more complex than in the previous section, for they involve the two operators q_1 and q_2, for which there are no closed expressions in general. We get:

Definition 4.2. The map $t \mapsto \Theta(t)$ for t such that $|t| \cdot \ell_{V_0} < 1$ defined by

$$\Theta_{11}(t) = q_2 + \alpha^* tZ(I - V_0^* tZ)^{-1}(\Lambda^{-1/2})^{(1)} q_1, \tag{4.6}$$

$$\Theta_{12}(t) = \alpha^*(I - tZV_0^*)^{-1}\Lambda^{-1}\beta(I+\beta^*\Lambda^{-1}\beta)^{-1/2}, \tag{4.7}$$

$$\Theta_{21}(t) = \beta^* tZ(I - tV_0^* Z)^{-1}(\Lambda^{-1/2})^{(1)} q_1, \tag{4.8}$$

$$\Theta_{22}(t) = (I + \beta^*(I - tZV_0^*)^{-1}\Lambda^{-1}\beta) \cdot (I+\beta^*\Lambda^{-1}\beta)^{-1/2} \tag{4.9}$$

will be called a rational J-inner function (in the non-stationary setting). If $\ell_{V_0} = 1$, then the function $t \mapsto \Theta(t)$ will be called 'of Brune type' or 'exhibiting Brune behaviour'.

Thus, the new ingredient is the parameter t. We note that $t \mapsto \Theta(t)$ is in fact analytic in the circle $|t| \cdot \ell_{V_0} < 1$. When α is invertible the above formulas take the simpler form

$$\begin{aligned} \Theta(t)_{11} = & \big(-\alpha^{-1}V_0\Lambda^{(1)} + \alpha^*(I - tZV_0^*)^{-1}tZ\big) \\ & \times\big(\Lambda^{(1)} + \Lambda^{(1)}V_0^*(\alpha\alpha^*)^{-1}V_0\Lambda^{(1)}\big)^{-1/2}, \end{aligned} \tag{4.10}$$

$$\Theta(t)_{12} = \alpha^*(I - tZV_0^*)^{-1}\Lambda^{-1}\beta(I+\beta^*\Lambda^{-1}\beta)^{-1/2}, \tag{4.11}$$

$$\Theta(t)_{21} = \beta^*(I - tZV_0^*)^{-1}tZ \cdot (\Lambda^{-1/2})^{(1)} \cdot (I + V_0^*(\alpha\alpha^*)^{-1}V_0)^{-1/2}, \quad (4.12)$$

$$\Theta(t)_{22} = (I + \beta^*(I - tZV_0^*)^{-1}\Lambda^{-1}\beta) \cdot (I + \beta^*\Lambda^{-1}\beta)^{-1/2}. \quad (4.13)$$

We also note that in general $t \mapsto \Theta(t)$ is a "mixed section", which contains a Brune and a Blaschke part; see Section 7 for more on this issue.

The terminology "*rational J-inner*" will be explained in the following section and in Theorem 6.1, which is the non-stationary analogue of Theorem 2.2.

It seems difficult to extend the theory to the case of indefinite metrics (i.e. when Λ would be merely invertible self-adjoint, but not necessarily positive) because of the square roots appearing in the formulas. This problem does not appear in the stationary case, where one can consider in a unified way the positive and non-positive cases.

4.2. The case $V_0 \in \mathbb{T}^{\mathbb{Z}}$

We study the special case where the entries are scalars and where, in the data defining the Brune section, $\alpha\alpha^* = \beta\beta^* = I$ and $V_0 \in \mathbb{T}^{\mathbb{Z}}$. Then Λ is a scalar diagonal operator.

Proposition 4.3. *Under the present assumptions, formulas* (3.12)–(3.15) *become*

$$\begin{aligned}&\big(\alpha(\Theta(t)_{11} - \Theta(t)_{12}(\Theta(t)_{22})^{-1}\Theta(t)_{21})\big)^{\wedge}(V_0)\\ &\quad = (t-1)V_0\frac{\Lambda^2 + \Lambda}{\Lambda + 1 - t\Lambda},\end{aligned} \quad (4.14)$$

$$\big(\alpha\Theta(t)_{12}\Theta(t)_{22}^{-1}\big)^{\wedge}(V_0) = \frac{\beta}{1 + (1-t)\Lambda}, \quad (4.15)$$

$$\big(\Theta(t)_{22}^{-1}\big)^{\wedge}(V_0) = \sqrt{\frac{1+\Lambda}{\Lambda}}\left(1 - \frac{1}{1+\Lambda}\sum_0^{\infty}\left(\frac{t\Lambda}{1+\Lambda}\right)^n \beta^{(n)*}\beta\right). \quad (4.16)$$

Proof. We prove only the previous formula. The proofs of the others are similar and will be omitted. Under the present assumptions we have

$$\begin{aligned}\big(\Theta(t)_{22}^{-1}\big)^{\wedge}(V_0) &= \sqrt{1 + \frac{1}{\Lambda}}\left(I - \frac{1}{1+\Lambda}\beta^*\left(\sum_0^{\infty}\left(\frac{t\Lambda}{1+\Lambda}\right)^n Z^n V_0^{[n]*}\right)\beta\right)\\ &= \sqrt{1 + \frac{1}{\Lambda}}\left(I - \frac{1}{1+\Lambda}\left(\sum_0^{\infty}\left(\frac{t\Lambda}{1+\Lambda}\right)^n Z^n \beta^{(n)*} V_0^{[n]*}\right)\beta\right)\\ &= \sqrt{1 + \frac{1}{\Lambda}}\left(I - \frac{1}{1+\Lambda}\sum_0^{\infty}\left(\frac{t\Lambda}{1+\Lambda}\right)^n Z^n V_0^{[n]*}\beta^{(n)*}\beta\right).\end{aligned} \quad (4.17)$$

Hence the result since $V_0^{[n]}V_0^{[n]*} = I$. $\square$

In particular we have the interpolation property:

$$\lim_{t\to 1}\left\|\left(\alpha\Theta(t)_{12}\Theta(t)_{22}^{-1}\right)^{\wedge}(V_0)-\beta\right\|_{\ell_2(\mathbb{Z})}=0. \tag{4.18}$$

On the other hand it does not hold in general that

$$\lim_{t\to 1}\Sigma(t)_{22}^{\wedge}(V_0)=0$$

since, in general

$$I\neq\sum_{0}^{\infty}\left(\frac{\Lambda}{1+\Lambda}\right)^{n}\cdot\frac{\beta^{(n)*}\beta}{1+\Lambda}. \tag{4.19}$$

For instance, take

$$\beta=\mathrm{diag}\left(\ldots,-1,\boxed{1},-1,1,\ldots\right).$$

Then $\beta^{(2n)}\beta=I$ while $\beta^{(2n+1)}\beta=-I$, and (4.19) is easily shown to hold.

4.3. *The stationary case*

In this section, we check that formulas (3.12)–(3.15) indeed reduce to the Redheffer transform of a Brune factor in the stationary case: setting $\alpha=1$, β, $v_0\in\mathbb{T}$ and Λ being a strictly positive number and replacing Z by the complex variable z we have

$$\Sigma_{11}(z)=\Lambda\left(-v_0+(\Lambda+1-zv_0^*\Lambda)^{-1}\right), \tag{4.20}$$

$$\Sigma_{12}(z)=\frac{\beta}{\Lambda+1-zv_0^*\Lambda}, \tag{4.21}$$

$$\Sigma_{21}(z)=\sqrt{1+\frac{1}{\Lambda}}\frac{z\beta^*\Lambda}{\Lambda+1-z\Lambda v_0^*}, \tag{4.22}$$

$$\Sigma_{22}(z)=\sqrt{1+\frac{1}{\Lambda}}\frac{\Lambda(1-zv_0^*)}{\Lambda+1-zv_0^*\Lambda}. \tag{4.23}$$

These are the entries of a Blaschke factor based on the point

$$\frac{v_0\Lambda}{\Lambda+1}$$

and the Redheffer transform of this factor is a Brune factor based on the point v_0^*. Indeed, we know from general principles that the (inverse) Redheffer transform of $\Sigma(z)$ is a J-inner rational function with a unique singularity. Since $(\Sigma_{22}(z))^{-1}$ has a pole at v_0, this transform is a Brune section at the point v_0. We leave the details to the reader.

5. Points of local losslessness

In this section, we follow the strategy described in Section 2.2 in the non-stationary case. We defined Brune sections as functions $t \mapsto \Theta(t)$ to avoid dealing with unbounded operators. In this spirit, we will find a representation of $S \in \mathscr{S}$ of the form

$$S(t) = \big(\Theta(t)_{11}\sigma(t) + \Theta(t)_{12}\big)\big(\Theta(t)_{21}\sigma(t) + \Theta(t)_{22}\big)^{-1},$$

where $\sigma \in \mathscr{S}$, $t \mapsto \Theta(t)$ is a rational J-inner function, and $S(t)$ and $\sigma(t)$ are the Zadeh extensions of S and σ.

Definition 5.1. Let $S \in \mathscr{S}$ and $\alpha \in \mathscr{D}$ with $\alpha\alpha^*$ non-singular. A diagonal operator V_0 with $\ell_{V_0} = 1$ will be called a point of local losslessness in the direction α if the following conditions hold:
1. Both Λ and Λ^{-1} are strictly positive diagonal operators, where

$$\Lambda = \sup_{0 \leqslant r < 1} \Lambda_r \tag{5.1}$$

with

$$\Lambda_r = \left(\sum_0^\infty r^{2n} V_0^{[n]} \left(\alpha\alpha^* - (\alpha S)^\wedge(rV_0)(\alpha S)^\wedge(rV_0)^*\right)^{(n)} V_0^{[n]*} \right). \tag{5.2}$$

2. The limit $\beta = \lim_{r \to 1} (\alpha S)^\wedge(rV_0)$ exists in the operator topology.

We note that Λ is a solution of

$$\Lambda - V_0 \Lambda^{(1)} V_0^* = \alpha\alpha^* - \beta\beta^*. \tag{5.3}$$

When α is scalar (i.e. for each diagonal entry k, k of the form $\alpha_k I$ with α_k scalar) and when V_0 is unitary, Λ_r can be rewritten as

$$\frac{\alpha\alpha^*}{1 - r^2} - \sum_0^\infty r^{2n} \left((\alpha S)^\wedge(rV_0)(\alpha S)^\wedge(rV_0)^*\right)^{(n)}.$$

If moreover $(\alpha S)^\wedge(rV_0)$ is a scalar diagonal operator, we get back to the stationary formula

$$\Lambda_r = \frac{I - \left((\alpha S)^\wedge(rV_0)\right)\left((\alpha S)^\wedge(rV_0)\right)^*}{1 - r^2}.$$

Theorem 5.2. *Let $S \in \mathscr{S}$ and let V_0 a point of local losslessness for S in the direction α. Then there exists a Brune section $t \mapsto \Theta(t)$, $0 \leqslant t < 1$, and a $\sigma \in \mathscr{S}$ such that*

$$S(t) = T_{\Theta(t)}(\sigma(t)). \tag{5.4}$$

Conversely, if $S(t)$ is given by (5.4) *in which $\Theta(t)$ is a Brune section, then S has a PLL in the direction α.*

In (5.4), we note that $S(t)$ and $\sigma(t)$ are the Zadeh extensions of S and σ, respectively, but $\Theta(t)$ is not the Zadeh extension of a bounded operator.

Proof of Theorem 5.2. Theorem 3.2 applied to $V = rV_0$ and $\beta = \beta_r = (\alpha S)^\wedge(rV)$ implies that for every r there exists a $\sigma_r \in \mathscr{S}$ such that $S = T_{\Theta_r}(\sigma_r)$, where Θ_r is obtained from formulas (3.2) with the present choice of V, α, β. By Theorem (3.6), $S(t) = T_{\Theta_r(t)}(\sigma_r(t))$, where

$$(\Theta_r)_{11}(t) = q_{2r} + \alpha^* tZ(I - \mathrm{tr}V_0^*Z)^{-1}(\Lambda_r^{-1/2})^{(1)}q_{1r}, \tag{5.5}$$

$$(\Theta_r)_{12}(t) = \alpha^*(I - \mathrm{tr}ZV_0^*)^{-1}\Lambda_r^{-1}\beta_r(I + \beta_r^*\Lambda_r^{-1}\beta_r)^{-1/2}, \tag{5.6}$$

$$(\Theta_r)_{21}(t) = \beta_r^* tZ(I - \mathrm{tr}ZV_0^*)^{-1}(\Lambda_r^{-1/2})^{(1)}q_{1r}, \tag{5.7}$$

$$(\Theta_r)_{22}(t) = (I + \beta_r^*(I - \mathrm{tr}ZV_0^*)^{-1}\Lambda_r^{-1}\beta_r) \cdot (I + \beta_r^*\Lambda_r^{-1}\beta_r)^{-1/2}, \tag{5.8}$$

and in which q_{1r} and q_{2r} have to be chosen in such a way that the block diagonal operator

$$\begin{pmatrix} (\Lambda_r^{1/2})^{(1)} rV_0^*(\Lambda_r + \beta_r\beta_r^*)^{-1/2} & q_{1r} \\ \alpha^*(\Lambda_r + \beta_r\beta_r^*)^{-1/2} & q_{2r} \end{pmatrix} \tag{5.9}$$

is unitary (it suffices for this that the respective diagonals of index k, k form unitary matrices). For the remaining argument it will be important that q_{1r} and q_{2r} be chosen in such a way that their (diagonally pointwise) limits exist and are equal to q_1 and q_2, where q_1 and q_2 are the values of q_{r1} and q_{r2} for $r = 1$, which certainly exist due to the hypothesis that S has a PLL at the diagonal point V_0 in the direction α. In fact, q_1 and q_2 are such that the block diagonal operator

$$\begin{pmatrix} (\Lambda^{1/2})^{(1)} V_0^*(\Lambda + \beta\beta^*)^{-1/2} & q_1 \\ \alpha^*(\Lambda + \beta\beta^*)^{-1/2} & q_2 \end{pmatrix} \tag{5.10}$$

is unitary. Let $a_{rk} = (\Lambda_r^{1/2})^{(1)} rV_0^*(\Lambda_r + \beta_r\beta_r^*)^{-1/2}$, and $b_{rk} = \alpha^*(\Lambda_r + \beta_r\beta_r^*)^{-1/2}$ and a, c the corresponding values for $r = 1$. Clearly $\lim_{r\to 1} a_{rk} = a_k$ and $\lim_{r\to 1} c_{rk} = c$ for each k, the question is whether the same can be asserted for the q's. Since $\binom{a_{rk}}{c_{rk}}$ is isometric,

$$\mathbf{P}_{rk} = I - \begin{pmatrix} a_{rk} \\ c_{rk} \end{pmatrix} \begin{pmatrix} a_{rk}^* & c_{rk}^* \end{pmatrix} \tag{5.11}$$

will be a projection operator, and there will be (for each k) continuity, $\lim_{r\to 1} \mathbf{P}_{rk} = \mathbf{P}_k$. A Gram–Schmidt decomposition of $\mathbf{P}_{rk}$ produces

$$\mathbf{P}_{rk} = \begin{pmatrix} q_{1rk} \\ q_{2rk} \end{pmatrix} \begin{pmatrix} q_{1rk}^* & q_{2rk} \end{pmatrix}. \tag{5.12}$$

The continuity of $q_{irk}, i = 1, 2$, is an easy property of the Gram–Schmidt orthogonalization procedure, well documented in the numerical literature. Entrywise convergence of a sequence of diagonal operators is equivalent to weak convergence (by the dominated convergence theorem). Now, let $\Theta_r(t)$ be the chain scattering matrix corresponding to the realization

$$\mathbf{M}_{\Theta_r}=\begin{pmatrix}(\Lambda_r^{1/2})^{(1)}V_r^*\Lambda_r^{-1/2} & q_{r1} & -(\Lambda_r^{1/2})^{(1)}V_r^*\Lambda_r^{-1}\beta_r(I+\beta_r^*\Lambda^{-1}\beta_r)^{-1}\\ \alpha^*\Lambda_r^{-1/2} & q_{r2} & \alpha^*\Lambda_r^{-1}\beta_r(I+\beta_r^*\Lambda_r^{-1}\beta_r)^{-1/2}\\ \beta_r^*\Lambda_r^{-1/2} & 0 & (I+\beta_r^*\Lambda_r^{-1}\beta_r)^{1/2}\end{pmatrix}, \tag{5.13}$$

in which we have used the continuous q_{1r} and q_{2r}. This realization corresponds to the formulas given above and which have been obtained from applying Theorem 3.2.

From the above, we can now ascertain weak convergence:

$$\lim_{r\to 1}(\Theta_r)_{11}(t)=q_2+\alpha^*Z(I-tV_0^*Z)^{-1}(\Lambda^{-1/2})^{(1)}, \tag{5.14}$$

$$\lim_{r\to 1}(\Theta_r)_{12}(t)=\alpha^*(I-tZV_0^*)^{-1}\Lambda^{-1}\beta(I+\beta^*\Lambda^{-1}\beta)^{-1/2}, \tag{5.15}$$

$$\lim_{r\to 1}(\Theta_r)_{21}(t)=\beta^*tZ(I-tV_0^*Z)^{-1}(\Lambda^{-1/2})^{(1)}q_1, \tag{5.16}$$

$$\lim_{r\to 1}(\Theta_r)_{22}(t)=(I+\beta^*(I-tZV_0^*)^{-1}\Lambda^{-1}\beta)\cdot(I+\beta^*\Lambda^{-1}\beta)^{-1/2}. \tag{5.17}$$

We set $\Theta(t)_{ij}=\lim_{r\to 1}(\Theta_r)_{ij}(t)$ where the limit has to be interpreted in the weak operator sense.

Now, may be via a subsequence, the weak limit w-$\lim_{r\to 1}\sigma_r$ exists (and is equal to σ, say). By Proposition 3.7, $\sigma_r(t)$ converges weakly to $\sigma(t)$ for every $t\in(0,1)$. We have

$$\sigma_r(t)=\big(S(t)(\Theta_r)_{21}(t)-(\Theta_r)_{11}(t)\big)^{-1}\big(-S(t)(\Theta_r)_{22}(t)+(\Theta_r)_{12}(t)\big),$$

and so

$$\big(S(t)(\Theta_r)_{21}(t)-(\Theta_r)_{11}(t)\big)\sigma_r(t)=\big(-S(t)(\Theta_r)_{22}(t)+(\Theta_r)_{12}(t)\big).$$

Hence, taking weak limits on both sides, we obtain

$$\big(S(t)(\Theta)(t)_{21}-\Theta(t)_{11}\big)\sigma(t)=\big(-S(t)\Theta(t)_{22}+\Theta(t)_{12}\big).$$

Since for $t\in(0,1)$ we have that $\Theta(t)$ is J-contractive (see Proposition 3.8), the map $(\Theta(t)_{21}\sigma(t)+\Theta(t)_{22})$ is invertible in $\mathscr{U}$ and hence

$$S(t)=T_{\Theta(t)}(\sigma(t)).$$

The converse of the theorem is obtained from a direct evaluation of $S(t)$ in function of $\Theta(t)$ and $\sigma(t)$. From the bilinear expression we have

$$-S(t)=\big(\Theta_{11}(t)\sigma(t)+\Theta_{12}(t)\big)\big(\Theta_{21}(t)\sigma(t)+\Theta_{22}(t)\big)^{-1}. \tag{5.18}$$

Subtracting $\Theta_{12}(t)\Theta_{22}^{-1}(t)$ and remarking that $\Theta_{11}(t)-\Theta_{12}\Theta_{22}^{-1}(t)\Theta_{21}(t)=\Sigma_{11}(t)$, we find for $-S(t)$

$$-S(t)=\Theta_{12}(t)\Theta_{22}^{-1}(t)+\Sigma_{11}(t)\sigma(t)\big(\Theta_{21}(t)\sigma(t)+\Theta_{22}(t)\big)^{-1}. \tag{5.19}$$

In these expressions, all inverses are causal and bounded. From the realization for Σ and using the normalized quantities $V_{0n}^* = (\Lambda^{1/2})^{(1)} V_0^* \lambda^{-1/2}$, $\alpha_n^* = \alpha^* \Lambda^{-1/2}$ and $\beta_n^* = \beta^* \Lambda^{-1/2}$, we find

$$\Sigma_{11} = q_2 + \alpha_n^*(I + \beta_n \beta_n^*)^{-1} \big(I - Z V_n^* (I + \beta_n \beta_n^*)^{-1}\big)^{-1} Z q_1. \tag{5.20}$$

Using $V_{0n} V_{0n}^* + \alpha_n \alpha_n^* = I + \beta_n \beta_n^*$ and $\alpha_n q_2 + V_n q_1 = 0$ we obtain

$$\alpha_n \Sigma_{11} = (Z - V_{0n}) \big(I - V_{0n}^* (I + \beta_n \beta_n^*)^{-1} Z\big)^{-1} q_1, \tag{5.21}$$

or, denormalizing,

$$\alpha \Sigma_{11} = (Z - V_0) \big(I - (\Lambda^{1/2})^{(1)} V_0^* (\Lambda + \beta\beta^*) Z\big)^{-1} q_1. \tag{5.22}$$

Let $m = \sigma(\Theta_{21}\sigma + \Theta_{22})^{-1}$ and $\Sigma_{12} = -\Theta_{12}\Theta_{22}^{-1}$. Then, from (5.19) we have

$$\alpha S = \alpha \Sigma_{12} - (Z - V_0) \big(I - (\Lambda^{1/2})^{(1)} V_0^* (\Lambda + \beta\beta^*) Z\big)^{-1} q_1 m, \tag{5.23}$$

where the Zadeh extension ‘(t)’ can be dropped because all the operators involved are bounded. It follows from the concatenation rule for the W-transform

$$(ST)^\wedge(V) = (S^\wedge(V) T)^\wedge(V)$$

that

$$\lim_{r\to 1} (\alpha S)^\wedge(r V_0) = \lim_{r \to 1} (\alpha \Sigma_{12})^\wedge (r V_0) - (1 - r)[\cdots] = \beta, \tag{5.24}$$

where $[\cdots]$ stands for a uniformily bounded diagonal. Let now $(\alpha S)^\wedge(r V_0) = \beta_r$. Then by the Nevanlinna–Pick theorem we have that there exists a σ_r such that

$$S(t) = T_{\Theta_r(t)}(\sigma_r(t)),$$

where Θ_r is given by (5.5) (from the existence of Θ follows the existence of Λ_r as a bounded invertible operator). Proceeding as before on the data $\{V_{0r}^*, \alpha, \beta_r\}$, we find

$$\alpha(\Sigma_r)_{11} = \alpha(Z - r V_0) \left[I - (\Lambda_r^{1/2})^{(1)} r V_0^* (\Lambda_r + \beta_r \beta_r^*) Z \right]^{-1} q_{1r}. \tag{5.25}$$

Hence $((\Sigma_r)_{11})^\wedge(r V_0) = 0$ and it follows, again by the concatenation rule for W-transforms

$$(\alpha S)^\wedge(r V_0) = (T_{\Theta_r}(\sigma_r))^\wedge(r V_0) = (\alpha \Sigma_{12})^\wedge(r V_0).$$

That $(\alpha S)^\wedge(r V_0) = (\alpha \Sigma_{12})^\wedge(r V_0)$ satisfies the definition for a PLL now follows easily by direct evaluation using the realization for Σ_{12}. □

In the stationary case, Eq. (5.2) reduces to

$$\Lambda_r = \frac{1 - |S(r V_0)|^2}{1 - r^2}$$

and one gets back the results of [7].

6. Reproducing kernel spaces associated to *J*-inner rational sections

We prove the analogue of Theorem 2.2 in the present setting. To simplify the exposition, we assume also here that α is invertible.

Theorem 6.1. *Let $\alpha, \beta, V_0 \in \mathscr{D}$ with α invertible and $\ell_{V_0} = 1$ and let*

$$F(t) = \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} (I - tZV_0^*)^{-1}, \quad t \in \mathbb{D}. \tag{6.1}$$

We assume that for $D \in \mathscr{D}$,

$$F(t)D \equiv 0 \implies D = 0$$

(*controllability hypothesis*) *and that Eq.* (5.3) *has a strictly positive and invertible solution Λ. Then, for $W \in \mathscr{D}$ with $\ell_W < 1$,*

$$F(t)\Lambda^{-1}\left(F(t)^\wedge(W)\right)^* = \left(J - \Theta(t)J\,(\Theta(t))^\wedge\,(W)^*\right)(I - t^2 ZW^*)^{-1}$$

if Λ is a solution of the Eq. (3.7).

Corollary 6.2. *The linear span $\mathscr{M}$ of functions of the form $F(t)D$ with $D \in \mathscr{D}_2$, endowed with the inner product*

$$\left[F(t)D, F(t)D\right]_{\mathscr{M}} = D^*\Lambda D$$

is a reproducing kernel space with reproducing kernel

$$\left(J - \Theta(t)J\,(\Theta(t))^\wedge\,(W)^*\right)(I - t^2 ZW^*)^{-1}.$$

This theorem is proved for V_0 with $\ell_{V_0} < 1$ in [8, Theorem 4.1, pp. 177–179]. There $t = 1$ since there is no need of the Zadeh extension. The analysis of [8] remains valid when one considers Zadeh extensions.

For the following lemma in the setting of diagonal operators with $\ell_V < 1$, see [8, Theorem 4.1, p.177].

Lemma 6.3. *Let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be given by (3.9) *and let*

$$G(t) = C(I - tZA)^{-1}, \tag{6.2}$$

where t is in a small enough neighbourhood of the origin. Then, for $W \in \mathscr{D}$ with $\ell_W < 1$,

$$G(t)\left((G(t))^\wedge\,(W)\right)^* = \left(J - \Theta(t)J\left(\left(\Theta(t)^\wedge(W)\right)^*\right)\right)\left(I - t^2 ZW^*\right)^{-1}. \tag{6.3}$$

Proof. With the present notation, we have $\Theta(t) = D + G(t)tZB$. We take t such that $(I - tZA)$ is invertible in $\mathscr{U}$. Then

$$\begin{aligned}
(\Theta(t))^\wedge(W) &= D + (G(t)tZ)^\wedge(W)B \\
&= D + \left((G(t))^\wedge(W)tZ\right)^\wedge(W)B \\
&= D + \left(Z((G(t))^\wedge(W))^{(1)}\right)^\wedge(W)B \\
&= D + tW(G(t)^\wedge(W))^{(1)}B.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Theta(t)J\,(\Theta(t))^\wedge(W)^* &= (D + G(t)tZB)\,J\left(D^* + B^*t\left(G(t)^\wedge(W)\right)^{(1)*}W^*\right) \\
&= DJD^* + G(t)ZtBJD^* + DJB^*t\left(G(t)^\wedge(W)\right)^{(1)*}W^* \\
&\quad + G(t)tZBJB^*t\left(G(t)^\wedge(W)\right)^{(1)*}W^* \\
&= J - CC^* - tG(t)ZAC^* - tCA^*\left(G^\wedge(W)\right)^{(1)*}W^* \\
&\quad + G(t)t^2Z\left(G^\wedge(W)\right)^{(1)*} - G(t)t^2ZAA^*\left(G^\wedge\right)^{(1)*}W^* \\
&= J - \mathbf{1} - \mathbf{2} - \mathbf{3} + \mathbf{4} - \mathbf{5},
\end{aligned}$$

where we follow the strategy and notation of [8]. But it is easily seen that

$$\begin{aligned}
\mathbf{1} + \mathbf{2} &= G(t)C^*, \\
\mathbf{3} + \mathbf{5} &= tG(t)A^*\left(G(t)^\wedge\right)(W)^{(1)*}W^*
\end{aligned}$$

and so

$$\begin{aligned}
\Theta(t)J\Theta(t)^\wedge(W)^* = J &- G(t)C^* - tG(t)A^*G(t)^\wedge(W)^{(1)*}W^* \\
&+ t^2G(t)ZG(t)^\wedge(W)^{(1)*}W^*.
\end{aligned}$$

To conclude we use the fact that

$$G(t)^\wedge(W) = C + tWG(t)^\wedge(W)^{(1)}A.$$

This is proved for $t = 1$ in [8, p. 179] and the proof is the same for $t \in (0, 1)$. $\square$

The proof of Theorem 6.1 follows exactly, up to the Zadeh extension, the arguments of [8, pp. 177–178].

7. Separation of Brune and Blaschke parts

We start out this section by giving some simple but rather general unicity and factorization theorems. Next, we specialize the theory to the so called ‘IVI-case’, that is the case where the system is linear time-invariant both in $-\infty$ and $+\infty$ regions (possibly different).

Lemma 7.1. *If M_1 and M_2 are two minimal J-unitary realizations of the same Brune section $\Theta(t)$ and if the corresponding Redheffer transform is used* (*i.e. the A operator has $\ell_A < 1$*), *then M_1 and M_2 are unitarily equivalent in the sense that there exists a sequence of unitary state transformations $Q_k \cdots Q_{k+1}^{-1}$ such that*

$$\begin{pmatrix} Q_k & & \\ & I & \\ & & I \end{pmatrix} M_1 \begin{pmatrix} Q_{k+1}^{-1} & & \\ & I & \\ & & I \end{pmatrix} = M_2.$$

Proof. Corresponding to a Brune section $\Theta(t)$ there is a unique inner scattering operator Σ with realization given by the Redheffer transformation on realizations. Let these transformed realizations for M_i, $i = \{1, 2\}$, be given by m_i, and let

$$m_i = \begin{pmatrix} A^{[i]} & B^{[i]} \\ C^{[i]} & D^{[i]} \end{pmatrix}.$$

By the realization theory for bounded operators (see [9]) the block rows of $(I - A^{[i]}Z)^{-1}B^{[i]}$ form, for each k, an orthonormal basis for the finite dimensional observability or controllability space of Σ. Hence these bases are unitarily equivalent, i.e. there exists a sequence of unitary matrices $\{Q_k\}$ such that, with $Q = \mathrm{diag}[Q_k]$

$$\left(I - A^{[2]}Z\right)^{-1}B^{[2]} = Q\left(I - A^{[1]}Z\right)^{-1}B^{[1]}.$$

Hence

$$A_k^{[2]} = Q_k A_k^{[1]} Q_{k+1}^{-1} \quad \text{and} \quad B_k^{[2]} = Q_k B_k^{[1]}.$$

The property $C_k^{[2]} = C_k^{[1]} Q_{k+1}^{-1}$ follows readily from minimality. That the same state transformation now applies to the realizations of the J-unitary representation is also immediate from the Redheffer transform. $\square$

Given a specific J-unitary realization

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of a causal J-inner operator possibly exhibiting 'Brune behaviour' (i.e. such that for its transition operator A, $\ell_A = 1$) and with the Zadeh extension

$$\Theta(t) = D + CtZ(I - AtZ)^{-1}B$$

one may wonder how to decompose $\Theta(t)$ in a product of sections with lower state space dimensions, keeping the J-inner property. In particular, we may attempt to factor $\Theta(t)$ as

$$\Theta(t) = \Theta_1(t)\Theta_2(t)\Theta_3(t)$$

in which $\Theta_1(t)$ and $\Theta_3(t)$ are of 'Brune type' while $\Theta_2(t)$ is of Blaschke type. We shall see in the following section that at least in one important special case, such a factorization is indeed possible. In the present section, we show how the realization can be factored in elementary sections. We follow and extend the treatment given in [9]. For numerical reasons and without impairing generality, we always use sections in which $D_{21} = 0$, and hence we assume the realization

$$M = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & D_{22} \end{array}\right). \tag{7.1}$$

Furthermore, we take M to be J-unitary, in the sense that

$$M^* J_1 M = J_2, \quad M J_2 M^* = J_1$$

for appropriate J_i's of the form

$$J_i = \left(\begin{array}{c|cc} I & & \\ \hline & I & \\ & & -I \end{array}\right),$$

where the dimensions of the individual blocks may be different. So in general $J_1 \neq J_2$. The local operators

$$M_k = \left(\begin{array}{c|cc} A_k & B_{1,k} & B_{2,k} \\ \hline C_{1,k} & D_{11,k} & D_{12,k} \\ C_{2,k} & 0 & D_{22,k} \end{array}\right)$$

at stage k are such that the submatrices

$$\mathbf{A}_k \stackrel{\text{def.}}{=} \left(\begin{array}{c|c} A_k & B_{1,k} \\ \hline C_{1,k} & D_{11,k} \end{array}\right)$$

and $D_{22,k}$ are square.

Finally, we assume that the transition operator A decomposes as

$$\begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \tag{7.2}$$

in which the entries are (of course) diagonal operators, or, dually, a (diagonal block) lower form, according to some recipe (for motivation see the following proposition). In [9, Chapter 14], it is shown that there always exist state transformations $Q_k \cdots Q_{k+1}^{-1}$ that put each A_k in upper echelon form. With the transition operator of a Θ in such a form, M now takes the form

$$\left(\begin{array}{cc|cc} A_1 & A_{12} & B_{11} & B_{12} \\ 0 & A_2 & B_{21} & B_{22} \\ \hline C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & 0 & D_{22} \end{array}\right) \tag{7.3}$$

and M is J-unitary as well, for the J_i, $i = 1, 2$, defined earlier in this section. Let now

$$M_1 = \left(\begin{array}{cc|cc} A_1 & & B'_{11} & B'_{12} \\ & I & 0 & 0 \\ \hline C_{11} & 0 & D'_{11} & D'_{12} \\ C_{21} & 0 & 0 & D'_{22} \end{array}\right)$$

be a J-unitary completion of the first block-column of (7.1)—which is always possible to manufacture, e.g., through Jacobi and hyperbolic rotations, see in particular [9, Chapter 9], Then we can also find doubly accented quantities such that M factors as $M = M_1 M_2$ with

$$M_2 = \left(\begin{array}{cc|cc} I & & 0 & 0 \\ 0 & A_2 & B_{21} & B_{22} \\ \hline 0 & C''_{12} & D''_{11} & D''_{12} \\ 0 & C''_{22} & 0 & D''_{22} \end{array}\right)$$

also J-unitary for the appropriate J. This amounts to a sketch for the proposition.

Proposition 7.2. *Suppose*

$$M = \left(\begin{array}{cc|c} A_1 & A_{12} & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right)$$

is a realization of a causal J-inner operator $\Theta(t)$. Then M can be factored as $\Theta(t) = \Theta_1(t)\Theta_2(t)$, where $\Theta_1(t)$ and $\Theta_2(t)$ have realizations

$$M_1 = \begin{pmatrix} A_1 & B'_1 \\ C_1 & D'_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C''_2 & D''_2 \end{pmatrix},$$

$D = D'_1 D''_2$ is a non-singular factorization of D,

$$B'_1 = B_1 (D''_2)^{-1}$$
$$C''_2 = (D'_1)^{-1} C_2.$$

and the realizations M_1 and M_2 are J-unitary.

The proof of the proposition is classical [9].

7.1. *The IVI case*

An important special case is when the system is time-invariant both in the region where $k \to -\infty$ and $k \to \infty$ (for possibly different systems), while being time-varying in between, the so-called IVI case, for 'invariant-varying-invariant'. For this case, we can give a strong factorization theorem, which also applies to LTI systems as a special instance.

Proposition 7.3. *Suppose that M is a J-unitary realization for a causal, J-inner $\Theta(t)$ which is such that*

$$M_\infty = \left(\begin{array}{cc|c} U_\infty & A_{12,\infty} & * \\ 0 & A_{22,\infty} & * \\ \hline * & * & * \end{array}\right)$$

in which U_∞ is a square matrix with all its eigenvalues on the unit circle. Then $\Theta(t)$ exhibits Brune behaviour. Likewise if $\Theta(t)$ is such that

$$M_{-\infty} = \left(\begin{array}{cc|c} U_{-\infty} & 0 & * \\ A_{21,-\infty} & A_{22,-\infty} & * \\ \hline * & * & * \end{array}\right)$$

in which $U_{-\infty}$ has all its eigenvalues of magnitude one, then $\Theta(t)$ exhibits Brune behaviour.

Proof. The proof is similar for the two cases, we suffice with case 1. We have to show that $\ell_A = 1$ for the A belonging to the realization for $\Theta(t)$. Looking at the transitional product for an arbitrary n

$$A^{\{n\}} = AA^{(-1)} \cdots A^{(-n+1)}$$

and specializing to k so large that it lays in the $+\infty$-LTI zone of $\Theta(t)$, we find

$$A_k^{\{n\}} = A_k \cdots A_{k+n-1} = \begin{pmatrix} U_\infty^n & * \\ 0 & * \end{pmatrix}$$

and hence $\|A^{\{n\}}\| \geqslant 1$. Therefore $\ell_A = \lim_{n\to\infty} \|A^{\{n\}}\|^{1/n} = 1$, since also $\ell_A \leqslant 1$ by the causality assumption. $\square$

In the case that the transition operator A_∞ in M_∞ has eigenvalues on the unit circle, we are entitled to say that $\Theta(t)$ exhibits Brune behaviour at $+\infty$. Similarly, if $A_{-\infty}$ has eigenvalues on the unit circle, $\Theta(t)$ will exhibit Brune behaviour at $-\infty$. The following theorem shows, among other things, that a locally finite chain scattering operator of IVI type, which does not exhibit Brune behaviour neither at $+\infty$ nor $-\infty$, must necessarily be of Blaschke type. We are now ready for the main factorization theorem of this section.

Theorem 7.4. *Let $\Theta(t)$ be a causal J-inner operator of the IVI-type. Then $\Theta(t)$ can be factored as*

$$\Theta(t) = \Theta_1(t) \cdot \Theta_2(t) \cdot \Theta_3(t),$$

where $\Theta_1(t)$ is of Brune type at $+\infty$, $\Theta_2(t)$ is of Blaschke-type, and $\Theta_3(t)$ is of Brune type at $-\infty$.

Proof. Starting out with a J-unitary realization for $\Theta(t)$, we may find an orthonormal transformation $Q_\infty \cdots Q_\infty^{-1}$ such that the transition matrix in the transformed realization has the form

$$\begin{pmatrix} U_\infty & A_{12,\infty} \\ 0 & A_{22,\infty} \end{pmatrix}$$

in which U_∞ has all its eigenvalues of magnitude one, and those of $A_{22,-\infty}$ are all strictly less than one in magnitude. Furthermore, we can determine state transformations $Q_k \cdots Q_{k+1}^{-1}$ for all k such that the block upper triangular Jordan form is maintained for all k. This is achieved, in a stable numerical manner, by recursively determining Q_k, assuming knowledge of Q_{k+1}, so that $Q_{k+1} \cdot A_k Q_{k+1}^{-1}$ is upper triangular (upper echelon form). Using the factorization theorem given above we can now produce

$$\Theta(t) = \Theta_1(t)\Theta_1'(t)$$

in which $\Theta_1(t)$ has U_∞ as transition matrix at $+\infty$, and the transition matrix of $\Theta_1'(t)$ is $A_{22,\infty}$. Proceeding dually on $\Theta_1'(t)$ but now with respect to $-\infty$, we find

$$\Theta_1'(t) = \Theta_2(t)\Theta_3(t),$$

where $\Theta_3(t)$ exhibits Brune behaviour, but now at $-\infty$. $\Theta_2(t)$ has realizations at $-\infty$ and $+\infty$ whose transition matrices have eigenvalues strictly less that one. It follows now immediately from the sprectral radius formula that this is the realization of a bounded, J-inner operator Θ_3. Hence it is of Blaschke type. □

Looking at the details of the factorization, the remark that a large collection of factorizations should be possible seems obvious. The theorem just given provides only one of the possibilities, a more refined study might indicate in which cases left and right Brune sections could be combined.

8. Conclusions

The present paper completes the representation theory of J-inner operators, the J-unitary operators that correspond to inner operators via a Redheffer transform. These operators can be unbounded, corresponding to what is known classically as *Brune sections*. It turns out that numerically, these J-inner sections can be realized much in the same way as is the case with the classical Blaschke sections, yet they represent unbounded operators. Essential in the computation is the existence of a positive definite solution to the Lyapunov–Stein equation. This equation, as well as the realizations can be recursively solved, which amounts in finite calculations when the original system is finitely specified. The trick that allows for the representation of the unbounded operators is the Zadeh extension, here generalized to the linear time-varying context.

A major application of the present theory is in time-varying H_∞ control, i.e. control for least sensitivity. Following the methodology of Kimura [14] one is given a ‘chain operator’ $G(\epsilon)$ depending on a gain parameter ϵ, and one wishes to know necessary and sufficient condition for factorization of $G = \Theta G_0$ into a general J-inner operator Θ and an outer operator G_0. While the extraction of a Brune section would not impact on the outerness of G_0, it would greatly enhance the chances that G_0 would be boundedly invertible, since the resulting Θ could take care of the unboundedness. This would then result in a much more attractive structure for least-sensitivity feedback, because actually the Redheffer transformation of Θ is actually used in the feedback structure, and it will be uniformily exponentially stable, as shown in the paper. The theory would then lead to a much stronger factorization theorem for least-sensitivity feedback purposes. This part of the theory lies outside the scope of the present paper and remains to be done.

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